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VARIATIONAL PROBLEMS WITH A SMALL PARAMETER IN THE THEORY OF ELASTICITY

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This study is a continuation of [1]. The latter examined a variant of the Signorini problem and boundary-value problems of the theory of elasticity for a cylinder with a corrugated lateral surface undergoing rapid oscillation. Here, we examine a variant of the Signorini problem in which a restriction is imposed on the lateral surface of a cylindrical region and the stresses are equal to zero on the planes x_3 = const. This variant is of interest due to its noncoercive nature. To a certain extent, the results presented here reinforce the findings in [2]. In the study of a body with a rapidly vibrating lateral surface, it is assumed that the boundary conditions are chosen so as to have the body be a cylinder compressed in a rigid ring. It is shown that rapid oscillation of the boundary leads to a situation whereby the limiting problem in the boundary conditions turns out to be the factor Γ − the ratio of the length of the undisturbed part of the boundary to the length of the disturbed portion. The boundary conditions on the lateral surface should be fairly specific, unlike the case (for example) of zero displacement of this surface.

1. Let us recall the formulation of the problem from [1]. We will examine an elastic, transversely isotropic cylinder $Q = \omega \times (-h/2, h/2)$, where ω is a finite region on a plane with a fairly smooth boundary γ. Let E be the elastic modulus in the plane of isotropy of the material $x_3 \equiv$ const, and let E′ be the elastic modulus in the orthogonal plane. We take the square root of the ratio E/E′ as the small parameter. In a physically realistic situation, ε would be small for an elastic cylinder reinforced in the direction of the vertical axis by a set of boron or carbon fibers having an elastic modulus significantly higher in the axial direction than in the circumferential direction.

We divide the stresses by the elastic modulus E, keeping the previous notation for the dimensionless stresses, and we use Hooke's law to express the stresses through the strains:

$$\sigma_{11} = a_{11}e_{11} + a_{12}e_{22} + a_{13}e_{33},\ \sigma_{12} = 2(1+\nu)^{-1}e_{12},$$
$$\sigma_{22} = a_{12}e_{11} + a_{11}e_{22} + a_{13}e_{33},\ \sigma_{13} = 2b\varepsilon^{-2}e_{13},$$
$$\sigma_{33} = a_{13}(e_{11} + e_{22}) + a_{33}\varepsilon^{-2}e_{33},\ \sigma_{23} = 2b\varepsilon^{-2}e_{23}.$$

Here

$$a_{11} = (1 - \mu^2\varepsilon^2)(1+\nu)^{-1}a_0^{-1};\ \ a_{12} = (\nu + \mu^2\varepsilon^2)(1+\nu)^{-1}a_0^{-1};$$
$$a_{13} = \mu a_0^{-1};\ \ a_{33} = (1-\nu)a_0^{-1};\ \ a_0 = 1 - \nu - 2\mu^2\varepsilon^2;$$
$$e_{ij} = 2^{-1}(u_{i,j} + u_{j,i})(i,j = 1,2,3, u_{i,j} = \partial u_i/\partial x_j)$$

(summation is performed from 1 to 3 over repeating indices); b = E′/G′ is the ratio of the elastic modulus E′ to the shear modulus G′ in the direction orthogonal to the plane of isotropy; ν is the Poisson's ratio in the plane of isotropy; μ is the auxiliary Poisson's ratio; u_k (k = 1, 2, 3) are the displacements. It follows from the positiveness of the potential strain energy that $0 < \nu < 1$, $a > 0$, $b > 0$.

We studied a mixed problem for the system of equations of the theory of elasticity in [1]

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$$-\sigma_{ij,j} + f_i = 0,\ f_i \in L^2(Q),\ i = 1,2,3,$$
$$\sigma_{i3}|_{x_3} = \pm h/2 = 0,\ u_i|_{\gamma \times (-h/2, h/2)} = 0$$

where it was shown that the solution converges slowly in H^1 to the problem of the tension-compression and bending of an isotropic plate. In order to subsequently account for the dependence of the solution on the parameter ε, we will denote the stresses and displacements by the superscript ε. Here, σ_{ij}^0 and u_i^0 will represent the stresses and displacements in the limiting problem. We put

$$a^\varepsilon(u^\varepsilon, v) = 2^{-1}\int_Q \sigma_{ij}(u^\varepsilon)e_{ij}(v)dx.$$

2. We will examine a variant of the Signorini problem with a certain restriction on the lateral surface of the cylinder. Let a closed convex cone K_1 exist in $W = [H^1(Q)]^3$:

$$K_1 = \{u \in W;\ u_n \leqslant 0 \text{ on } S = \gamma \times (-h/2, h/2)\}.$$

We will study the asymptotic behavior of the variational inequality

$$a^\varepsilon(u^\varepsilon, v - u^\varepsilon) \geqslant (f, v - u^\varepsilon)\ \forall\, v \in K_1. \tag{2.1}$$

as $\varepsilon \to +0$. The quadratic form $a^\varepsilon(u^\varepsilon, u^\varepsilon)$ is noncoercive on K_1. In order for the minimum of the functional

$$J(u^\varepsilon) = a^\varepsilon(u^\varepsilon, u^\varepsilon) - F(u^\varepsilon),\ F(u^\varepsilon) = \int_Q f_k u_k^\varepsilon dx$$

to exist on K_1 in accordance with theorem 10.1 in [3], the following condition must be satisfied

$$F(\rho) \leqslant 0\ \forall\, \rho \in R',\ R' = R \cap K_1 \tag{2.2}$$

(R is the space of rigid-body displacements). If condition (2.2) is satisfied exactly, i.e. if the sign of the inequality remains unchanged when and only when $\rho \in R^*$ (where R^* is a subset of R' formed by the bilateral displacements, i.e. displacements such that ρ and $-\rho$ are compatible with the constraints on the body), then $J(u^\varepsilon)$ has an absolute minimum on K_1. We will henceforth require the use of theorem 1.4 from [4].

Theorem. Let $\|u\|'$ be a half-norm on a Hilbert space H,

$$K = \{u \in H;\ \|u\|' = 0,\ \dim R < \infty\}.$$

We assume that

$$C_1\|u\| \leqslant \|u\|' + \|P_R u\| \leqslant C_2\|u\|$$

(P_R is an operator allowing orthogonal projection on R). Let K be a closed convex subset and let $\mathscr{P}$ be a penalty operator. We assume that the differential of $\mathscr{P}$ is positively uniform, i.e. that $D\mathscr{P}(tu, h) = tD\mathscr{P}(u, h)$ for any $t > 0$. Let $f \in H$, $K \cap R \neq \{0\}$ u — $(f, h) > 0$ when $h \in K \cap R$, $h \neq 0$. Then the following inequality is valid

$$\|u\|'^2 + \mathscr{P}(u) - (f, u) \geqslant C_1\|u\| - C_2.$$

Let us consider passing to the limit at $\varepsilon \to +0$ in inequality (2.1). As in Part 2 in [1], we relate (2.1) to a problem with a penalty:

$$a^\varepsilon(u^{\varepsilon\eta}, v) + \eta^{-1}\int_S [u_n^{\varepsilon\eta}]^+ v_n dS = \int_Q fv dx. \tag{2.3}$$

Its solution is determined to within the field of rigid-body displacements:

$$\rho_1 = a + \gamma x_2 - \beta x_3, \rho_2 = b - \gamma x_1 + \alpha x_3, \rho_3 = c + \beta x_1 - \alpha x_2.$$

We then have the following estimate for problem (2.3)

$$a^\varepsilon(u^{\varepsilon\eta},u^{\varepsilon\eta}) \geqslant C\|u^{\varepsilon\eta}\|^2_W - C_2.$$

In fact, if we put

$$\mathscr{P}(u) = 2^{-1}\int_S([u^{\varepsilon,\eta}]^+)^2dS,\ (f, u) = \int_\Omega f_k u_k dx,$$

$$\|u\|' = (\int_\Omega e_{ij}(u)e_{ij}(u)dx)^{1/2},$$

then the above theorem makes it possible to obtain the required estimate, since estimates (1.6) from [1] remain valid in this case. Thus, we can isolate from the sequence $u^{\varepsilon,\eta}$ another sequence that converges slowly in W to the element $u^{0,\eta}$ (the same notation as was used previously is used here for the new sequence). Due to the compactness of the traces in $L^2(S)$, the term with the penalty converges to the expression

$$h\eta^{-1}\int_\gamma [g_1^\eta n_1 + g_2^\eta n_2]^+ (\psi_1 n_1 + \psi_2 n_2)ds + \frac{h^3}{12}\frac{1}{\eta}\int_\gamma \left(\frac{\partial u_3^{0,\eta}}{\partial n}\right)^+ \frac{\partial v_3}{\partial n}ds.$$

Here, ds is an element of arc length. The conditions of solvability of the initial problem (2.1) are transformed in this case: on S we have the inequality

$$\rho_k n_k|_S \leqslant 0,$$

it following from this that $\alpha = \beta = 0$ in the formulas for the rigid-body displacements (due to the arbitrariness with respect to x_3). Then

$$\rho_1 = a + \gamma x_2, \rho_2 = b - \gamma x_1, \rho_3 = c.$$

Having integrated over x_3 in the solvability condition

$$\int_\Omega (f_1\rho_1 + f_2\rho_2 + f_3\rho_3)dx \leqslant 0,$$

we find that the following is necessary for the limiting problem to be solvable

$$\int_\omega [\langle f_1\rangle(a + \gamma x_2) + \langle f_2\rangle(b - \gamma x_1) + c\langle f_3\rangle]\, dx \leqslant 0,$$

so that

$$\int_\omega [\langle f_1\rangle(a + \gamma x_2) + \langle f_2\rangle(b - \gamma x_1)]dx' \leqslant 0,$$

$$\int_\omega \langle f_3\rangle dx' = 0,\ dx' = dx_1 dx_2.$$

The passage to the limit at $\eta \to +0$ is based on well-established methods [5]. Here, the initial variational inequality is split in two: the first expression has the form

$$d(g,g - \psi) \geqslant (\langle f\rangle,g - \psi)\ \forall\ \psi \in K_2, \tag{2.4}$$

where K_2 is a closed convex cone in $[H^1(\omega)]^2$ generated by the condition $gn \mid \gamma \leq 0$, corresponding to a two-dimensional Signorini problem (the condition necessary for its solution is satisfied). The second inequality

$$b(u_3^{0,0}, v_3 - u_3^{0,0}) \geqslant (\langle f_3\rangle, v_3 - u_3^{0,0})\ \forall\, v_3 \in K_3. \tag{2.5}$$

Here, K_3 is a closed convex cone in $H^2(\omega)$ generated by the condition $\partial v_3/\partial n \mid \gamma \leq 0$; b(u, v), d(u, v) are bilinear symmetric forms consisting of the factors with h and h^3 in Eq. (1.9) from [1]. In (2.4)-(2.5), $\langle f\rangle$ is the mean of f over the thickness of the cylinder.

Let us now formulate the final result.

Theorem 2.1. At $\varepsilon \to +0$, the functions u_1^ε, u_2^ε are the solution of variational equation (2.1), converging slowly to the solution of variational inequality (2.4), while u_3^ε converges slowly to the solution of inequality (2.5).

The problem of solving inequality (2.5) was examined in [2], where it was proven to be solvable with the stronger assumptions

$$\int_\omega \langle f_3\rangle x_k dx' = 0,\ k = 1,2,\ \int_\omega \langle f_3\rangle dx' = 0$$

(ω is a bounded convex set with a regular boundary). The decrease in the number of solvability conditions is connected with the fact that the functions u_3^0 are determined to within a constant. It follows from theorem 2.1 that the convexity of the region is also not a necessary condition.

3. Let us examine one more variant of boundary conditions on the lateral surface of the cylinder. Let on S

$$\sigma_\tau = 0,\ \sigma_n + ku_n = 0,\ k > 0. \tag{3.1}$$

The mechanical interpretation of boundary conditions (3.1) [5] is as follows: the shear stress vanishes, while the normal forces are forces of elastic inertia and are proportional to the absolute value of the normal displacement. Let V_1 be a Hilbert space:

$$V_1 = \{v \in W;\ v_3|_S = 0,\ v_\tau|_S = 0\},$$
$$v_n = v_1 n_1 + v_2 n_2,\ v_\tau = -v_1 n_1 + v_2 n_1.$$

We put

$$a_2^\varepsilon(u,v) = a^\varepsilon(u^\varepsilon,v) + k\int_S u_n^\varepsilon v_n dS.$$

The variational problem consists of determining the function $u^\varepsilon \in W$, which satisfies the integral identity

$$a_2^\varepsilon(u^\varepsilon,v) = (f,v)\ \forall\, v \in V_1.$$

Analogously to theorem (4.2), we can prove from [2] that there exists a constant $\alpha > 0$ such that

$$a_2^\varepsilon(u^\varepsilon,u^\varepsilon) \geqslant \alpha \|u^\varepsilon\|_W^2 .$$

Thus, the given problem has a uniqe solution, and estimates (1.6) from [1] are valid for it. Passing to the limit at $\varepsilon \to 0$ in the term

$$k\int_S u_n^\varepsilon v_n dS,$$

after we integrate over x_3 we obtain the expression

$$kh\int_\gamma g_n\psi_n ds + \frac{kh^3}{12}\int_\gamma \frac{\partial u_3^0}{\partial n}\frac{\partial v_3}{\partial n}ds. \tag{3.2}$$

We put

$$n_{11} = \varepsilon_{11}(g) + \nu\varepsilon_{22}(g), n_{22} = \nu\varepsilon_{11}(g) + \varepsilon_{22}(g), n_{12} = 2(1 - \nu)\varepsilon_{12}(g),$$

$$M_1 = (1 - \nu)\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2} n_1 + \left(\frac{\partial^2 u_3^0}{\partial x_2^2} + \nu\frac{\partial^2 u_3^0}{\partial x_1^2}\right) n_2,$$

$$M_2 = -(1 - \nu)\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2} n_2 - \left(\frac{\partial^2 u_3^0}{\partial x_2^2} + \nu\frac{\partial^2 u_3^0}{\partial x_1^2}\right) n_1,$$

$$M_\tau = -M_1 n_2 + M_2 n_1.$$

Since $u_3^0 \in H_0^1(\omega)$, we can write the Green formula for u_3^0 in the form

$$b(u_3^0, v) = (< f_3 > , v) - \left(M_\tau, \frac{\partial v_3}{\partial n}\right)\gamma.$$

It follows from (3.2) that g_1, g_2 solve the following problem: determine g_1, $g_2 \in H^1(\omega)$ from the integral identity

$$d(u,\psi) + kh\int_\gamma g_n \psi_n d\gamma = (< f_i > , \psi_i) \;\forall\; \psi \in V_1^0, \tag{3.3}$$

$$V_1^0 = \{v \in [H^1(\omega)]^2, v_\tau = -v_1 n_2 + v_2 n_1|_\gamma = 0\}.$$

Here, u_3^0 is the solution of the problem

$$\begin{gathered} b(u_3^0, v_3) + k\frac{h^3}{12}\int_\gamma \frac{\partial u_3^0}{\partial n}\frac{\partial v_3}{\partial n} d\gamma = (< f_3 > , v_3) \\ \forall\; v_3 \in H_0^1(\omega) \cap H^2(\omega), M_\tau - k\partial u_3^0/\partial n|_\gamma = 0. \end{gathered} \tag{3.4}$$

The solutions of problems (3.2) and (3.3) are unique.

Theorem 3.1. At $\varepsilon \to +0$, u_1^ε and u_2^ε converge slowly in W to the solution of problem (3.3), while u_3^ε converges slowly to the solution of problem (3.4).

4. Let us examine a problem similar to that studied in Part 3, assuming that the lateral surface of the region is corrugated. A similar problem with a rapidly oscillating boundary was examined in [2] for the Laplace equation. It should be noted that boundary-value problems of the theory of elasticity for a transversely isotropic body with a corrugated lateral surface were studied in [6] by the method of regular boundary perturbation – in contrast to the method used in the present investigation.

Let $Q_\varepsilon = \omega_\varepsilon \times (-h/2, h/2)$, with ω_0 being a bounded region on a plane having a smooth boundary $\partial\omega_0$ and an outer unit normal N. We use s to designate the curvilinear abscissa with the curve $\partial\omega_0$. In the neighborhood of $\partial\omega_0$, s and N are curvilinear coordinates on a plane. We will examine the smooth periodic function $y_2 = F(y_1)$ with the period 1 in the rectangular coordinates y_1, y_2. We define the boundary $\partial\omega_\varepsilon$ of region ω_ε by the equation $N = \varepsilon \times F(x/\varepsilon)$ (where ε is a small positive parameter). We assume that the small parameter, characterizing the oscillation of the boundary, coincides with the small parameter that characterizes the anisotropy of the body. This assumption is not essential and is made only to simplify the notation.

We will consider the situation in which there are no singular perturbations in the system of equations of the theory of elasticity ($\varepsilon = 1$ in the generalized Hooke's law). As in Part 3, we put

$$V_1 = \{v \in W;\, v_3|_{S_\varepsilon} = 0,\, v_\tau|_{S_\varepsilon} = 0,\, S_\varepsilon = \partial\omega_\varepsilon \times (-h/2, h/2)\}.$$

The vector-function u^ε is determined from the integral identity

$$a^\varepsilon(u^\varepsilon, v) + k\int_{S_\varepsilon} u_n^\varepsilon v_n dS_\varepsilon = (f, v) \;\forall\; v \in V_1. \tag{4.1}$$

As in [2], we define the "waviness factor" Γ as the ratio of the length $\partial\omega_\varepsilon$ to the length $\partial\omega_0$.

Theorem 4.1. The solution of problem (4.1) converges slowly in $[H^1(Q_0)]^3$ to the solution of the problem

$$a(u^0,v) + k\Gamma\int_{S_0} u_n^0 v_n dS_0 = (f,v)\ \forall\, v \in V_1,$$
$$S_0 = \partial\omega_0 \times (-h/2,h/2).$$

The proof of this theorem is analogous to the proof of theorem 8.1 in [2] and is omitted here.

Now let us examine the situation in which a singular perturbation exists in the system of equations. In this case, the term

$$k\int_{S_\varepsilon} u_n^\varepsilon v_n dS_\varepsilon$$

in Eq. (4.1) reduces to the following in accordance with lemma 8.1 from [2]

$$kh\Gamma\int_\gamma g_n \psi_n d\gamma_0 + k\Gamma\frac{h^3}{12}\int_\gamma \frac{\partial u_3^0}{\partial n}\frac{\partial v_3}{\partial n} d\gamma_0$$

and the limiting problems for determining u_3^0, g_1, and g_2 have the form (3.2)-(3.3), where the multiplier Γ should be placed in front of the integral of γ.

Theorem 4.2. When the system of equations contains a singular perturbation in the limit at $\varepsilon \to +0$, the solution of problem (4.1) splits into the solutions of the problems:

$$d(g,\psi) + kh\Gamma\int_\gamma g_n\psi_n d\gamma = (\langle f_i\rangle, \psi_i)\ \forall\, \psi \in V_1^0, \tag{4.2}$$
$$V_1^0 = \{v \in [H^1(\omega)]^2,\ v_\tau = -v_1 n_2 + v_2 n_1|_\gamma = 0\};$$

$$b(u_3^0,v_3) + k\frac{h^3}{12}\Gamma\int_\gamma \frac{\partial u_3^0}{\partial n}\frac{\partial v_3}{\partial n} d\gamma = (\langle f_3\rangle, v_3)$$
$$\forall\, v_3 \in H_0^1(\omega) \cap H^2(\omega),\ M_\tau - k\Gamma\, \partial u_3^0/\partial n|_\gamma = 0. \tag{4.3}$$

We should make several comments here regarding the behavior of the results obtained above. It follows from Eqs. (4.2) and (4.3) that the boundary conditions of the initial problem are transformed in the passage to the limit. The boundary conditions of the limiting problem contain the multiplier Γ, which is connected with the fact that the boundary conditions of the initial problem correspond to a cylinder compressed in a rigid ring. The boundary is straightened during deformation, which in turn leads to a change in the boundary conditions. Such a change in boundary conditions would not be possible if, for example, we assigned the displacements on the lateral surface.

REFERENCES

1. Yu. A Bogan, "Certain variational problems with a small parameter in the theory of elasticity," Prikl. Mat. Mekh., **49**, No. 4, 604-607 (1985).
2. É. Sanchez-Palencia, Inhomogeneous Media and the Theory of Vibration [Russian translation], Mir, Moscow (1983).
3. G. Fikera, Existence Theorems in the Theory of Elasticity [Russian translation], Mir, Moscow (1974).
4. J. Necas and I. Hlavacek, Mathematical Theory of Elastic and Elasto-plastic Bodies: an Introduction, Elsevier, Amsterdam (1981).
5. G. Duveaux and J.-L. Lyons, Inequalities in Mechanics and Physics [Russian translation], Mir, Moscow (1980).
6. A. N. Guz' and Yu. N. Nemish, Perturbation Methods in Three-Dimensional Problems of the Theory of Elasticity [in Russian], Vishcha Shk., Kiev (1982).